Math 1A: Calculus **Instructor: Alexander Paulin**

Handout: Descartes' Rule of Signs

Discussions 201, 203 // 2018-09-28

You've seen the rules for differentiating polynomials in lecture, so now we're going to apply them to explore a result known as *Descartes' rule of signs*. Disclaimer: you will not be tested on the contents of this handout in any way, but the techniques we use here should serve as useful practice with the derivative.

Throughout this worksheet we will be considering polynomial functions, which I will remind you are of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

for some whole number $n \geq 0$, called the *degree* of the polynomial. Here $a_n, \ldots, a_0$ are constants—the *coefficients* of the polynomial. We require that the leading coefficient a_n is nonzero.

Example 1. A constant function such as $f(x) = 4$ is a polynomial of degree zero. A linear function (with nonzero slope) such as $f(x) = 2x - 7$ is a polynomial of degree one.

In general, solving polynomial equations like $x^5 - 15x^3 + 7x + 1 = 0$ is extremely hard. Even counting the exact *number* of solutions is really hard. The two equations

$$2x^2 - 5x + 3 = 0$$
$$2x^2 - 5x + 4 = 0$$

have very similar coefficients, but the first one has two solutions while the second has none. For higher degree equations it's much more difficult to tell.

While it's often infeasible to calculate the exact number of solutions to a polynomial equation, it's a well-known fact that if f is a polynomial of degree $n \geq 1$, then the equation $f(x) = 0$ has *at most* n solutions in total—these solutions are called *roots* of f. Descartes' rule of signs can be viewed as an improvement upon this fact: it gives us an upper bound on the number of *positive* roots of f. Amazingly, this upper bound can be calculated quickly in your head, without any fiddling with numbers! Here's the statement of the rule of signs.

Theorem 1 (Descartes' rule of signs, which I will abbreviate as DRS)**.** *Let $f(x)$ be a polynomial of degree $n \geq 1$. Looking at the coefficients of f from left to right, count the number of times that the sign of the coefficient changes, and call this number S.*

Then the equation $f(x) = 0$ has at most S positive solutions.

A completely equivalent way of saying this is as follows: suppose f is a polynomial of degree $n \geq 1$ for which $f(x) = 0$ has R positive solutions. Then the coefficients of f must change sign at least R times.

Example 2. For the polynomial

$$f(x) = x^5 - 15x^3 + 7x + 1,$$

the sign changes twice (from $+1$ to -15, and then from -15 to $+7$), so $S = 2$. DRS states that $f(x) = 0$ has at most two positive solutions. My computer tells me that there are actually exactly two: approximately ≈ 0.759 and ≈ 3.808.

Exercise 1. Again, using $f(x) = x^5 - 15x^3 + 7x + 1$, what does DRS say about the number of *negative* roots of f?
(Hint: reflect across the y-axis.)

For the rest of this handout, we're going to try and understand why Descartes' rule of signs is true.

We'll start things off simple, and consider $f(x) = a_1 x + a_0$, where $a_1 \neq 0$. For such a function f, the equation $f(x) = 0$ has $\underset{\text{number}}{\underline{\hspace{2cm}}}$ solution(s). It has a positive solution if a_1 and a_0 have ________(the same // differing) sign. Compare this to what DRS tells you.

Now let's consider $f(x) = a_2 x^2 + a_1 x + a_0$, where we require $a_2 \neq 0$. This is a quadratic, and we know how to solve $f(x) = 0$ explicitly using the quadratic formula. But since this handout is about the so-called "rule of signs," we're going to try and say what we can using *only* the signs of the coefficients.

Exercise 2. Compute the derivative $f'(x)$ and fill in the blanks (in terms of a_2, a_1, a_0):

$$f'(x) = \underline{\hspace{1.5cm}}\, x + \underline{\hspace{1.5cm}}.$$

Exercise 3. In terms of the coefficients a_2, a_1, a_0, what is $f(0)$? What is $f'(0)$? On the graph of $y = f(x)$, what do these values describe?

Exercise 4. Again in terms of the coefficients a_2, a_1, a_0, when is it the case that $f'(x) = 0$ has a positive solution? That is to say: when is there a $c > 0$ for which $f'(c) = 0$?

Exercise 5. For each of the following requirements, come up with a quadratic $f(x) = a_2x^2 + a_1x + a_0$ satisfying the stated conditions, and sketch its graph.

(1) $f(0) > 0$, $f'(0) > 0$, and $f'(c) = 0$ for some $c > 0$.
(2) $f(0) > 0$, $f'(0) > 0$, and $f'(c) \neq 0$ for all $c > 0$.
(3) $f(0) > 0$, $f'(0) < 0$, and $f'(c) = 0$ for some $c > 0$.
(4) $f(0) > 0$, $f'(0) < 0$, and $f'(c) \neq 0$ for all $c > 0$.

For each of your quadratics f: how many positive solutions to $f(x) = 0$ are there?

Exercise 6. Now compare your results from the preceding problem to what DRS tells you!

Hopefully the preceding exercises have convinced you that DRS is true for linear and quadratic polynomials. But perhaps that isn't so impressive—after all, it's easy enough for us to explicitly compute the zeros of such functions.

Now suppose $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ is a cubic, with $a_3 \neq 0$. In this case, $f(x) = 0$ is very difficult to explicitly solve.

Exercise 7. Well, this is a derivative handout after all, so I'm going to ask you to compute $f'(x)$ again:

$$f'(x) = \underline{\hspace{1.5cm}}x^2 + \underline{\hspace{1.5cm}}x + \underline{\hspace{1.5cm}}.$$

Fill in the blanks in terms of a_3, a_2, a_1, a_0.

Let's illustrate DRS with an example. In what follows, we're going to consider the cubic

$$f(x) = x^3 + x^2 - 10x - 8,$$

for which we have

$$f'(x) = \underline{\hspace{6cm}}.$$

Note that $f'(x)$ is a quadratic, and—hopefully—we're already convinced that DRS works for quadratics. Specifically, DRS tells us that $f'(x) = 0$ has at most $\underset{\text{number}}{\underline{\hspace{2cm}}}$ positive solutions. (By the way, how does counting the sign changes in the coefficients of f' relate to counting the sign changes in the coefficients of f?)

Now, a solution to $f'(x) = 0$ tells us a location at which the tangent line to $y = f(x)$ has slope zero. In other words, f reaches either a peak or a valley at those points.

Draw a picture to convince yourself that, between any two zeros of a polynomial f, there must be either a peak or a valley in the graph of f. This is a special case of the *mean value theorem*, which we will cover later in the course.

From this, we conclude that $f(x) = 0$ can have at most two positive solutions (do you see why?). But actually, according to DRS, $f(x) = 0$ in fact has at most $\underset{\text{number}}{\underline{\hspace{2cm}}}$ positive solution(s).

The reason is that there's still some information we haven't used yet! $f(0)$ is ________(positive // negative) and $f'(0)$ is ________(positive // negative). Let r be the smallest positive root of f. Draw a picture and see if you can tell why, in the interval $(0, r)$, the graph of f must have a ________(peak // valley)!

Let's summarize:

- The coefficients of f, read from left to right, are $+1, +1, -10, -8$.
- The first three numbers, $+1, +1, -10$, are related to the coefficients of f'. In particular, they have the same signs as the coefficients of f'. Since we believe DRS for quadratics, we conclude that $f'(x) = 0$ has at most one positive solution, because there is one sign change in $+1, +1, -10$.
- This means that the graph of $y = f(x)$ has at most one peak or valley to the right of the y-axis.
- Now we use the fact that there is no sign change between the last two coefficients: -10 and -8. This tells us that $f(0)$ and $f'(0)$ have the same sign.
- Altogether, we conclude that there can be at most one positive solution to $f(x) = 0$, because if there were more, we would necessarily have more than one peak or valley to the right of the y-axis in the graph of $y = f(x)$. And indeed, that "at most one" exactly comes from the fact that there is one sign change in $+1, +1, -10, -8$. DRS works!

Exercise 8. What does DRS say about the number of negative solutions to $x^3 + x^2 - 10x - 8 = 0$?

Exercise 9. Consider the equation $x^7 - 12x^4 + 8 = 0$. Since the polynomial $x^7 - 12x^4 + 8$ has degree 7, we know that the stated equation has at most 7 solutions.

Use DRS to argue why we know the equation actually has at most 3 solutions. (There actually are three solutions: $\approx -0.891, 0.919, 2.270$.)